SELF-OSCILLATING FLOW IN THE BOUNDARY LAYER

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INTRODUCTION

Self-oscillations exist in a viscous fluid as an intermediate mode in the transition from parallel laminar to turbulent flow.

The conception of the origin of turbulence as a sequence of laminar flows displacing each other and being more and more complicated was developed by Landau [1]. Excitation of new degrees of freedom hence occurs because of the loss of stability of the initial simpler mode.

Periodic self-oscillations in the homogeneous variable are the first mode in this sequence replacing the plane-parallel stationary stream. The existence of such self-oscillations was proved as a solution of the Navier–Stokes equations, which branches off from the stationary solution by Yudovich [2].

The branching off occurs at Reynolds numbers Re corresponding to points of the neutral curve according to linear stability theory. A definite spatial period and frequency of the self-oscillations originating corresponds to each such point.

An investigation of the nonlinear stability of Poiseuille flow in a plane channel [3] showed that a self-oscillating mode which exists for $\mathrm{Re} < \mathrm{Re}_*$ branches off at the critical Reynolds number $\mathrm{Re}_* = 5772$ (determined by the nose of the neutral curve). On the other hand, energy estimates show [4] that any deviation from the plane-parallel mode for $\mathrm{Re} < 50$ will damp out monotonically with time (energy-wise) and experimental results indicate that turbulence is realized only for $\mathrm{Re} \gtrsim 1000$. Therefore, Poiseulle flow at low Reynolds numbers is stable to any perturbations of arbitrarily high initial energy (absolutely stable), is unstable with respect to infinitesimal perturbations (absolutely unstable), and is metastable in a definite range of Reynolds numbers ($1000 < \mathrm{Re} < 5772$, i.e., is stable to perturbations whose energy is below the threshold value, and unstable to perturbations with a sufficiently high initial energy).

Self-oscillations correspond to definite finite values of the perturbation energy in the domain of parallel flow metastability and are periodic stationary solutions, on the average, of the Navier–Stokes equations. But under these conditions the self-oscillating solutions are themselves unstable, and hence under sufficiently large perturbations the stream evolves further although no developed turbulent mode is built up. Since this evolution is spread out in time, it is difficult to observe experimentally.

In this sense, the flow in a boundary layer differs profitably from flow in a channel, since the transition to turbulence is spread out spatially downstream and all the stages can be followed in sequence.

Test data indicate [5] that the Blasius flow is replaced by a self-oscillating mode (Tollmien–Schlichting waves) downstream (for $\mathrm{Re} > \mathrm{Re}_*$ according to linear theory) in the flow around a flat plate. The amplitude of the self-oscillations gradually grows and an abrupt transition to the turbulent flow mode occurs for Reynolds numbers almost an order greater than Re_*.

The self-oscillating mode was observed most clearly in the Schubauer and Skramstad tests

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[6], where the oscillations were initiated by vibrating strips. The self-oscillations are even tracked in the "natural" transition on a flat plate under weak external turbulence conditions, as experimental investigations [7, 8] indicate. A frequency analysis of pulsating motion showed that in the background of a comparatively homogeneous energy frequency distribution in the initial noise the intensities start to grow downstream for frequencies from a band corresponding to the inner domain of the neutral stability curve. The intensity at these frequencies exceeds the mean external noise intensity severalfold and the band of isolated frequencies itself varies, being aligned under the neutral curve. These data indicate the existence of stable self-adjusting oscillations in a definite Reynolds number band. Within the framework of the plane-parallel approach, the amplitude and frequency are established as a result of nonlinear interaction.

This paper is devoted to a theoretical analysis of the conditions for origination and the character of these self-oscillations for boundary-layer flow with an imposed pressure gradient.

1. Let us examine flow along a wedge. The potential flow velocity at the wedge surface is proportional to the power of the distance from the apex

$$U_\infty = Cx^m.$$

In this case the boundary-layer equations for a viscous incompressible fluid admit of self-similar solutions [5] for the stream function

$$\Phi(x, y) = \sqrt{\frac{2}{m+1}\nu x U_\infty}\,\psi_0(\eta), \quad \eta = y\sqrt{\frac{m+1}{2}\frac{U_\infty}{\nu x}},$$

where x and y are the longitudinal and transverse coordinates, respectively, ν is the coefficient of kinematic viscosity, and $m = (x/U_\infty)/(dU_\infty/dx)$ is the flow-form parameter, related uniquely to the wedge apex angle. The values $m > 0$ correspond to a positive wedge angle, and the pressure gradient is hence negative. The values $m < 0$ correspond to flow in a diffusor, the pressure gradient is hence positive, and boundary-layer separation occurs for $m = -0.0904$.

The function $\psi_0(\eta)$ is a solution of the Falkner–Skan equation

$$\psi_0''' + \psi_0\psi_0'' + \frac{2m}{m+1}\left(1 - \psi_0'^2\right) = 0;$$

$$\psi_0 = \psi_0' = 0 \text{ for } \eta = 0; \ \psi_0' = 1 \text{ for } \eta = \infty.$$

If the x velocity component is referred to its value on the outer boundary of the boundary layer, then the velocity profile is given by the expression $U(\eta) = \psi_0'$, where the prime denotes differentiation.

The stationary-flow perturbations are analyzed within the framework of the plane-parallel approximation. This means that for each Reynolds number defined as $\mathrm{Re} = (U_\infty \delta_*/\nu)$ (where δ_* is the displacement thickness), the flow is approximated by a one-dimensional parallel stream with velocity profile $U_{(y)}$. The transverse velocity component is assumed zero. Within the framework of this approximation, the perturbed motion stream function satisfies the equation

$$\frac{\partial \Delta\psi}{\partial t} + U\frac{\partial \Delta\psi}{\partial x} - U''\frac{\partial \psi}{\partial x} - \frac{1}{\mathrm{Re}}\Delta\Delta\psi = \frac{\partial \psi}{\partial x}\frac{\partial \Delta\psi}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \Delta\psi}{\partial x} \tag{1.1}$$

(Δ is the Laplace operator) with boundary conditions of adhesion to the wall and the requirement of minimum growth as $y \to \infty$. Use of the plane-parallel approximation induces a definite error in the analysis of origination of the self-oscillations. However, it may be hoped that the plane-parallel approximation will permit a qualitatively true description of the branching off of the self-oscillating solutions.

The linearized equation (1.1) has the spectrum of solutions periodic in x and t which is given by the neutral curve. Following [3], let us select an arbitrary point on the neutral curve corresponding to the Reynolds number Re_0, let us give the Reynolds number an increment

$$\mathrm{Re}=\mathrm{Re}_0+\varepsilon^2 f,\quad f=\pm 1$$

and let us seek the solution (1.1) as a series in the small parameter ε:

$$\psi=\frac{1}{\mathrm{Re}}\sum_{k=1}^{\infty}\varepsilon^k\psi_k(x-ct,y);\qquad \mathrm{Re}\,c=\mathrm{Re}_0\sum_{k=0}^{\infty}c_k\varepsilon^k. \tag{1.2}$$

Substituting (1.2) in (1.1) and equating the coefficients of ε^k to zero, we obtain the chain of equations

$$\Delta\Delta\psi_k-\mathrm{Re}_0\left[(U-c_0)\frac{\partial\Delta\psi_k}{\partial x}-U''\frac{\partial\psi_k}{\partial x}\right]=f\left[U\frac{\partial\Delta\psi_{k-2}}{\partial x}-U''\frac{\partial\psi_{k-2}}{\partial x}\right]+\sum_{j=1}^{k-1}\left[\left(\frac{\partial\psi_{k-j}}{\partial y}-\mathrm{Re}_0c_{k-j}\right)\frac{\partial\Delta\psi_j}{\partial x}-\frac{\partial\psi_{k-j}}{\partial x}\frac{\partial\Delta\psi_j}{\partial y}\right]. \tag{1.3}$$

For k = 1 the member in the expansion (1.2) has the form

$$\psi_1(x-ct,\ y)=\beta\ \{\varphi(y)\exp[i\alpha(x-ct)]+\bar{\varphi}(y)\exp[-i\alpha(x-ct)]\}$$

(the bar denotes the complex conjugate), where the complex amplitude $\varphi(y)$ is a solution of the Orr–Sommerfeld equation

$$L_\alpha\varphi\equiv\varphi^{IV}-2\alpha^2\varphi''+\alpha^4\varphi-i\alpha\mathrm{Re}[(U-c_0)(\varphi''-\alpha^2\varphi)-U''\varphi]=0 \tag{1.4}$$

with the boundary conditions $\varphi=\varphi'=0$ at y = 0 and the requirement that $\varphi(y)$ decreases at infinity.

It can be seen that the damping of $\varphi(y)$ as $y\to\infty$ will be exponential in nature. Indeed, it can be assumed that U" = 0 and U = 1 for sufficiently high y. Then (1.4) becomes an equation with constant coefficients and its solutions, which damp out as $y\to\infty$, are

$$\varphi=C_1\exp(-\alpha y)+C_2\exp(-\gamma y), \tag{1.5}$$

where

$$\gamma=\gamma_r+i\gamma_e;\quad \gamma_r=\sqrt{\frac{a+\sqrt{a^2+b^2}}{2}};\quad \gamma_e=\frac{b}{2\gamma_r},$$

$$a=\alpha^2,\quad b=\alpha\mathrm{Re}_0(1-c_0).$$

The exponential nature (1.5) of the damping of $\varphi(y)$ can be reflected in the form of the boundary conditions

$$(\varphi''-\alpha^2\varphi)'+\gamma(\varphi''-\alpha^2\varphi)=0;$$
$$(\varphi''-\gamma^2\varphi)'+\alpha(\varphi''-\gamma^2\varphi)=0,\quad y=A, \tag{1.6}$$

which should be substituted for sufficiently large y. A finite range of values $0<y<A$ was considered in a numerical computation of the problem. The boundary conditions (1.6) were posed for y = A. The computations were carried out for increasing values of A until the results ceased to change. Thus when A doubled from 5 to 10, no noticeable discrepancy was detected.

For k = 2 the number in the expansion (1.2) for the stream function is

$$\psi_2=\beta^2\ \{V_0(y)+V_1(y)\exp[2i\alpha(x-ct)]+\bar{V}_1(y)\exp[-2i\alpha(x-ct)]\}.$$

The function $V_1(y)$ should satisfy the boundary-value problem (1.4) with α replaced by 2α and the introduction of the inhomogeneity

$$L_{2\alpha}V_1 = i\alpha(\varphi'^2 - \varphi\varphi'')'.$$

The function $V_0(y)$ is subject to the equation $V_0'' = i\alpha(\bar{\varphi}\varphi' - \varphi\bar{\varphi}')$ and the conditions $V_0 = V_0' = 0$ at $y = 0$. Let us note that the function $V_0(y)$ grows linearly as $y \to \infty$, although the right side of the equation damps out exponentially. But the stream function for the initial flow ψ_0 also grows linearly, hence the ratio between the perturbed stream function and ψ_0 will remain small. The derivatives $\psi_2(x, y, t)$, which yield a contribution to the pulsating velocity field, hence remain finite. The condition of solvability of (1.3) for $k > 1$ is that c_k equal zero for odd k.

It follows from (1.3) that ψ_3 will be the sum of harmonic oscillations with wave numbers α and 3α. The condition of solvability of the equation for the amplitude of the first harmonic will be

$$-c_2\mathrm{Re}_0 I_1 + \beta^2 I_2 + f I_3 = 0;$$

$$I_1 = \int_0^\infty \theta(y)(\varphi'' - \alpha^2\varphi)\,dy; \quad I_2 = \int_0^\infty \theta(y)\,[V_0'(\varphi'' - \alpha^2\varphi) -$$

$$- V_1'(\bar{\varphi}'' - \alpha^2\bar{\varphi}) - 2V_1(\bar{\varphi}''' - \alpha^2\bar{\varphi}') - \varphi V_0'' + 2\bar{\varphi}'(V_1'' - 4\alpha^2 V_1) + \bar{\varphi}(V_1''' - 4\alpha^2 V_1')]\,dy; \qquad (1.7)$$

$$I_3 = \int_0^\infty \theta(y)\,[U(\varphi'' - \alpha^2\varphi) - U''\varphi]\,dy.$$

Here $\theta(y)$ is the solution of the problem conjugate to (1.4). Integration is over the infinite doman but the integrals converge because of the exponential damping of the integrands. The sign in (1.7) is selected such that the requirement $\beta^2 > 0$ is satisfied.

2. Numerical computations were carried out using the method of differential factorization with a splice [9]. The neutral stability curve 1 for Blasius flow ($m = 0$) is shown in Fig. 1. The arrows indicate where self-oscillating modes exist, the length of the arrows correspond qualitatively to the quantity $\partial E/\partial \mathrm{Re}$, and their direction to the sign of this derivative. Here E is the self-oscillation energy and on the neutral curve

$$\frac{\partial E}{\partial \mathrm{Re}} = f\beta^2 \int_0^\infty (|\varphi'|^2 + \alpha^2|\varphi|^2)\,dy.$$

On the lower branch of the neutral curve $\partial E/\partial \mathrm{Re} > 0$, i.e., self-oscillations exist for Reynolds numbers greater than Re_0 in the domain of initial flow instability. On the upper branch the self-oscillations branch off into the domain outside the neutral curve where the initial stream is stable. The change in the sign of $\partial E/\partial \mathrm{Re}$, which passes through zero at the point $\alpha = 0.356$, Re = 774 has no value in principle, but is related to the change in orientation of the stability domain relative to the neutral curve.

The critical Reynolds number for Blasius flow is $\mathrm{Re}_* = 519$ for $\alpha_* = 0.304$. Somewhat above this point on the neutral curve $\partial E/\partial \mathrm{Re}$ becomes infinite and changes sign at $\alpha = 0.31$. This corresponds to a change in the mode of self-oscillation origination. It is important to note that $\partial E/\partial \mathrm{Re} > 0$, at the nose of the neutral curve, i.e., the self-oscillations branch off towards high Reynolds numbers and are stable.

The nature of the branching off at the nose of the neutral curve is of value in principle. If the excitation is hard, i.e., the self-oscillations branch off in the range of lesser values of Re, then the flow is metastable for Reynolds numbers less than the critical according to linear theory. In this case, a nonlinear critical Reynolds number less than the linear Re_* exists for which self-oscillations of finite amplitude (or turbulence) can at once originate first. Curve 1 in Fig. 2 reflects this situation.

If the excitation is soft, i.e., $\partial E/\partial \mathrm{Re} > 0$, then for small perturbation amplitudes a stable self-oscillating mode exists which can be observed experimentally. The most typical situation in this case corresponds to curve 3 in Fig. 2. But the case described by curve 2 can even be realized. If a family of straight lines corresponding to values of the derivative

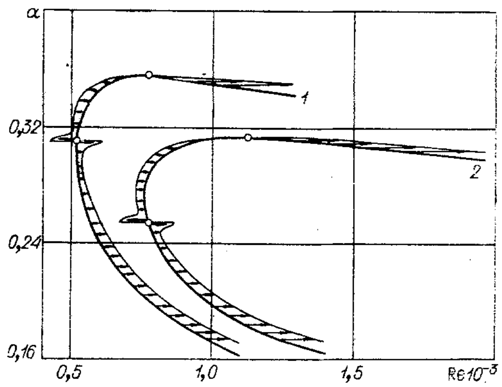

Fig. 1

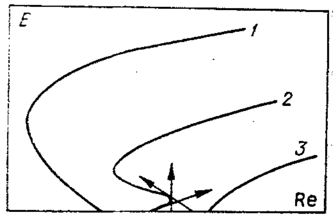

Fig. 2

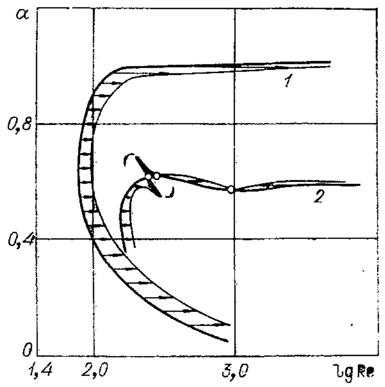

Fig. 3

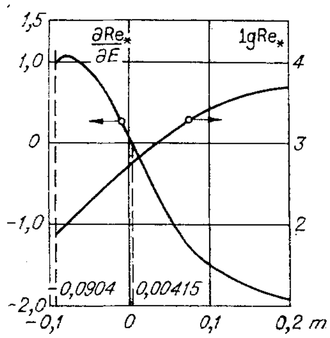

Fig. 4

$\partial E/\partial \mathrm{Re}$ at the points of the neutral curve is superposed, then curve 2 will be the envelope of this family for small amplitudes. Then in a small range $\mathrm{Re}>\mathrm{Re}_*$ stable small self-oscillations exist but they are metastable as is the initial flow. The nonlinear critical Reynolds number can hence be less than Re_*.

Such a case is most probable when the point at which $\partial E/\partial \mathrm{Re}$ changes sign is close to the critical point on the netural curve and lies somewhat above it, as is realized for Blasius flow. It can be assumed that the situation described by curve 2 in Fig. 2 holds here, where the critical values of the perturbation energy are small.

The fact that Blasius flow is the boundary case in the sense of hard or soft perturbations of the oscillations is shown by comparing it with gradient flows. The case of a negative pressure gradient at m = 0.0192 is represented by curve 2 in Fig. 1. The critical parameters are $\alpha_* = 0.272$, $\mathrm{Re}_* = 759$, and the point of the pole for $\partial E/\partial \mathrm{Re}$ is located at $\alpha = 0.254$, Re = 778. Therefore, the nose of the neutral curve lies in the hard excitation zone.

The situation is the reverse for a positive gradient. The values m = –0.0882 and –0.0602 correspond to curves 1 and 2 in Fig. 3. As the pressure gradient increases the pole point of $\partial E/\partial \mathrm{Re}$ shifts towards higher Re along the upper branch and, finally, vanishes by merging with the root of $\partial E/\partial \mathrm{Re}$ for m close to the separation value. For $m=-0.0882$ the excitation is soft on the whole neutral curve ($\mathrm{Re}_*=78.4$; $\alpha_*=0.662$).

Superposed in Fig. 4 as a function of m is $\partial \mathrm{Re}_*/\partial E$ at the point of the nose of the neutral curve. The hard nature of the excitation is realized for negative pressure gradients at $m>0.00415$.

It is interesting to note that the hard nature of the excitation is reinforced as the critical Reynolds number increases (see Fig. 4). This indicates that the nonlinear critical Reynolds number is more conservative than the linear relative to changes in the external parameters. The increase in Re_* is associated with the greater filling of the velocity profile. As in the case of a Hartmann boundary layer or an asymptotic velocity profile with suction, this results in the possibility of delaying the laminar mode to higher Reynolds numbers on one hand, and to explosive-like excitation of the turbulent mode for sufficiently large perturbations, on the other.

The critical Reynolds numbers are reduced significantly with the imposition of a positive pressure gradient, but the nature of the excitation of self-oscillations becomes soft. Here the situation is quite similar to the free convection case. After the loss of stability by the initial flow, a secondary laminar oscillating mode develops. Such oscillations are often observed behind poorly streamlined bodies. They precede the appearance of Kármán streets and the development of turbulence.

Therefore, the analysis conducted of the excitation of self-oscillations agrees completely with the phenomena observed in experiments.

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